2025 WAMC 10 Solutions

1. What is the value of
$$\frac{2025+|2025-(2025-2025)|-2025}{2025}?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 5 **(E)** 2025

Bruce Shu

Solution: Simplifying, we have
$$\frac{2025+|2025-(2025-2025)|-2025}{2025}=\frac{2025+|2025-0|-2025}{2025}$$
$$=\frac{2025+2025-2025}{2025}=\frac{2025}{2025}=\boxed{\textbf{(B)}\ 1}.$$

2. Points A, B, and C lie on circle ω with center O. If $\angle OAB=\angle OAC=20^\circ$, find $\angle OBC$.

(A) 10° **(B)** 20° **(C)** 40° **(D)** 50° **(E)** 70°

Ryan Tang

Solution: We know that $\angle BAC=20^\circ+20^\circ=40^\circ$, so $\angle BOC=2\angle BAC=80^\circ$. Because $OB=OC$, triangle BOC is isosceles, so $\angle OBC=\angle OCB=\frac{180^\circ-80^\circ}{2}=\boxed{\textbf{(D)}\ 50^\circ}$.

3. In a game, Alex has a perfectly spherical apple. He first places down a basic sprinkler, which increases the radius of the sphere by 50 percent. Then, he places down an advanced sprinkler, which further increases the radius by 100 percent. What is the percent increase of the volume of the new apple compared to the original?

(A) 2600 **(B)** 2700 **(C)** 2800 **(D)** 2900 **(E)** 3000

Alexander Yang

Solution: After the first sprinkler is placed, the radius is multiplied by 1.5. After the second sprinkler is placed, the radius is further multiplied by 2, so the total increase is a multiplication by $1.5\cdot 2=3$. The volume increases proportionally to the cube of the radius, so it gets multiplied by 27, which means the volume was increased by $\boxed{\textbf{(A)}\ 2600}\%$.

4. Arjun, Vincent, Jacob, and Alexander stand in a line. Arjun refuses to stand in the first or second position. Vincent refuses to stand in the second or third position. Jacob refuses to stand in the third or fourth position. Alexander refuses to stand in the fourth or first position. How many possible orderings are there?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Kaiyuan Mao

Solution: We casework based on where Arjun stands.
If he stands in the third position, then Alexander has to stand in the second position, so Jacob has to stand in the first position, and Vincent has to stand in the fourth position. This is one solution.
If Arjun stands in the fourth position, then Vincent has to stand in the first position, so Jacob has to stand in the second position, and Alexander has to stand in the third position. This is another solution.
We've exhausted all possible positions for Arjun, so the answer is $\boxed{\textbf{(C)}\ 2}$.

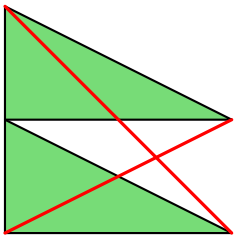
5. What is the maximum amount of intersections between 2 circles and 1 triangle?

   (A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

   Ryan Tang

Solution: The two circles can only intersect each other twice. Between a triangle and a circle, there is a maximum of 6 intersections. The answer is thus $2 + 6 + 6 = \boxed{\textbf{(E)}\ 14}$.

6. In the figure below, the sum of the squares of the lengths of the red segments is 10. The two congruent light green right triangles have leg lengths in the ratio $1 : 2$. Find their total area.



   (A) $\frac{6}{7}$ **(B)** $\frac{15}{13}$ **(C)** $\frac{9}{7}$ **(D)** $\frac{10}{7}$ **(E)** $\frac{20}{13}$

   Ryan Tang

Solution: Let the leg lengths of the green triangles be a and $2a$. Then, the shorter line segment has length $\sqrt{a^2 + (2a)^2} = \sqrt{5a^2}$, and the longer line segment has length $\sqrt{(2a)^2 + (2a)^2} = \sqrt{8a^2}$. The sum of their squares is $13a^2 = 10$, so $a^2 = \frac{10}{13}$.
The area of the two triangles is $2a^2 = \boxed{\textbf{(E)}\ \frac{20}{13}}$.

7. Ryan writes down the number 11111122. Patrick randomly rearranges the digits into a new eight-digit number. What is the probability that the sum of the digits in the left half equals the sum of the digits in the right half?

   (A) $\frac{1}{4}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{4}{7}$ **(E)** $\frac{5}{7}$

   Ryan Tang

Solution: Note that for the two sides to have equal sums, both halves must contain 2 exactly once each. Let us place a 2 randomly. Without loss of generality, assume it lands in the first half. Then, there are 4 spots to place the next 2 in the other half, and there are 7 spots remaining. Therefore, the answer is $\boxed{\textbf{(D)}\ \frac{4}{7}}$.

8. Squares $ABCD$, $DEFG$, and $GHIA$, each of side length 3, are drawn such that each square has no overlap with triangle ADG. Suppose each vertex of hexagon $BCEFHI$ lies on a circle. What is the area of the circle?

(A) $12\pi - 3\pi\sqrt{3}$ **(B)** 12π **(C)** $12\pi + 3\pi\sqrt{3}$ **(D)** $12\pi + 6\pi\sqrt{3}$ **(E)** $12\pi + 9\pi\sqrt{3}$

Wesley Wu

Solution: By symmetry, the center O of the circle is the center of the equilateral triangle as well. Consider just ADG and $ABCD$. Let the feet of the perpendiculars from O to AD and BC be P and Q, respectively. Note that the radius of the circle is equal to $OB = \sqrt{BQ^2 + OQ^2}$, and the area is $(BQ^2 + OQ^2)\pi$.
We already know that $BQ = \frac{3}{2}$, and to compute OQ, we simply need to find $OP + PQ = \frac{\sqrt{3}}{2} + 3$. Then, our answer is $\left(\frac{9}{4} + \frac{3}{4} + 9 + 3\sqrt{3}\right)\pi = \boxed{\textbf{(C)}\ 12\pi + 3\pi\sqrt{3}}$.

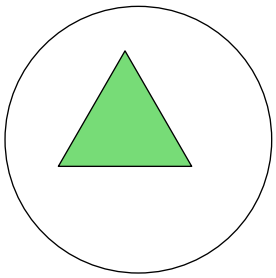
9. Let p be a prime for which there exist positive integers a and b satisfying $\sqrt{a^2 + p} = b^2$. Find the sum of the four smallest possible values of p.

(A) 104 **(B)** 116 **(C)** 126 **(D)** 138 **(E)** 144

Ryan Tang

Solution: Squaring both sides and rearranging yields $p = b^4 - a^2 = (b^2 - a)(b^2 + a)$. Since p is a prime, $b^2 - a$ must equal 1, and hence $a = b^2 - 1$ and $p = b^2 + a = 2b^2 - 1$. Then, simply testing values of b and checking for primality allows us to determine that $b = 2, 3, 4, 6$ result in $p = 7, 17, 31, 71$, which sum to $\boxed{\textbf{(D)}\ 126}$.

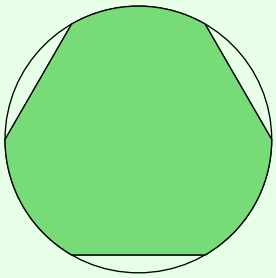
10. Laura takes a sponge in the shape of an equilateral triangle with side length 2 and dips it in paint. Then, without rotating the sponge or painting outside the circle, she paints as much of the interior of a circle with radius 2 as she can. How much of the circle is left unpainted?



(A) $2\pi - 3\sqrt{3}$ **(B)** $2\pi - 3\sqrt{2}$ **(C)** $2\pi - 2\sqrt{3}$ **(D)** $4\pi - 3\sqrt{3}$ **(E)** $4\pi - 2\sqrt{3}$

Laura Wang

Solution: Through observation, note that the following shaded region is the region of every part of the circle that can be painted.



Each of the small unshaded areas is created by the subtraction of an equilateral triangle from a sector. Each sector has central angle $60°$, so each one has area $\frac{2}{3}\pi$; each equilateral triangle has side length 2 and area $\sqrt{3}$. Thus, our answer is $3\left(\frac{2}{3}\pi - \sqrt{3}\right) = \boxed{\textbf{(A)}\ 2\pi - 3\sqrt{3}}$.
Note: We really should have put $4\pi - \sqrt{3}$ as a solution since it would have baited the many people who failed to read the question.

11. Ryan places a standard fair 6-sided die (with opposite faces summing to 7) on the table with the 1 face on top. He then rolls the die to an adjacent face three times in succession. What is the probability that no number greater than 3 ever appears on the top?

(A) $\frac{1}{216}$ **(B)** $\frac{1}{64}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{6}$

Ryan Tang

Solution: Each turn, there are four options to roll the die to. Note that the three numbers that satisfy the condition, 1, 2, and 3, can all be reached from the other two. As we start on 1, for each step, there are 2 valid moves of the total 4. The probability of this happening three times in a row is $\left(\frac{1}{2}\right)^3 = \boxed{\textbf{(D)}\ \frac{1}{8}}$.

12. Five cubes with side lengths 2, 3, 1, 5, 4 are stacked in that order, where the centers of touching sides align. What is the surface area of that tower (including the bottom face)?

(A) 270 **(B)** 275 **(C)** 278 **(D)** 286 **(E)** 330

Laura Wang

Solution: Let us count the lateral faces first. There are no overlaps, so the total surface area of the lateral faces is $4(1^2 + 2^2 + 3^2 + 4^2 + 5^2) = 220$.
Then, we must count the surface area of the horizontal faces. Consider two adjacent cubes with side lengths a and b, with $a > b$. The overlapping area would be $b^2 - a^2$. Therefore, our total horizontal area is $2^2 + (3^2 - 2^2) + (3^2 - 1^2) + (5^2 - 1^2) + (5^2 - 4^2) + 4^2 = 66$, so our total is $\boxed{\textbf{(D)}\ 286}$.

13. What is the area formed by all coordinate pairs (x, y) that satisfy

$$(x - x_1)^2 + (y - y_1)^2 \leq 4$$

over all x_1 and y_1 that satisfy $x_1^2 + y_1^2 = 200$?

(A) $80\pi\sqrt{2}$ **(B)** $100\pi\sqrt{2}$ **(C)** 200π **(D)** 204π **(E)** $160\pi\sqrt{2}$

Alexander Yang

Solution: For each (x_1, y_1) on the circle $x_1^2 + y_1^2 = 200$, the condition shades the disk of radius 2 centered at (x_1, y_1). The union of those disks is the annulus of all points whose distance r from the origin satisfies

$$\sqrt{200} - 2 \le r \le \sqrt{200} + 2.$$

The area is thus

$$\pi\left((\sqrt{200}+2)^2 - (\sqrt{200}-2)^2\right) = \pi \cdot 8\sqrt{200} = 8\pi\sqrt{200} = \boxed{\textbf{(A)}\ 80\sqrt{2}\pi}.$$

14. Four consecutive natural numbers a, b, c, d satisfy $a < b < c < d$. If a is divisible by 3, b is divisible by 5, c is divisible by 7, and d is divisible by 9, find the minimum value of $a + b + c + d$.

(A) 82 **(B)** 106 **(C)** 510 **(D)** 642 **(E)** 654

Kaiyuan Mao

Solution: Let $a = n$. The four conditions can be rewritten as

$$n \equiv 0 \pmod 3,\ \ n+1 \equiv 0 \pmod 5,\ \ n+2 \equiv 0 \pmod 7,\ \text{and}\ \ n+3 \equiv 0 \pmod 9.$$

The congruence $n+3 \equiv 0 \pmod 9$ has solution $n \equiv 6 \pmod 9$, which already implies $n \equiv 0 \pmod 3$, so we may instead work with the system

$$n \equiv -1 \equiv 4 \pmod 5,\ \ n \equiv -2 \equiv 5 \pmod 7,\ \text{and}\ \ n \equiv 6 \pmod 9.$$

Now, let $n = 9k + 6$. The first condition gives us $9k + 6 \equiv 4 \pmod 5 \implies k \equiv 2 \pmod 5$, and the second yields $9k + 6 \equiv 5 \pmod 7 \implies k \equiv 3 \pmod 7$. The third is satisfied through our substitution. By combining the two, we obtain $k \equiv 17 \pmod{35}$, so the minimum k is 17, and the minimum n is 159. Extracting, our answer is $159 + 160 + 161 + 162 = \boxed{\textbf{(D)}\ 642}$.

15. Real numbers x and y satisfy the system of equations

$$\begin{aligned} x^3 - 8x^2 + px - q &= 0, \\ y^3 - 8y^2 + py - q &= 0, \text{ and} \\ (x+y)^3 - 8(x+y)^2 + p(x+y) - q &= 0. \end{aligned}$$

There exists a unique k for which $kp + q$ is constant for all values of x and y. Find the value of $kp + q$.

(A) -64 **(B)** -58 **(C)** -32 **(D)** -10 **(E)** 26

Ryan Tang

Solution: Note that x, y, and $x + y$ are all roots of $n^3 - 8n^2 + pn - q$. Then, $x + y + (x + y) = 8$ by Vieta's, so $x + y = 4$. Then, plugging this into the last equation gives $-64 = (-4)p + q$, so $k = -4$ and $kp + q = \boxed{\textbf{(A)}\ -64}$.

16. Triangle $\triangle ABC$ intersects circle ω at points D, E, F, G, H, I such that $ADEB$, $BFGC$, and $CHIA$ are collinear in those orders. Given that $AD = 3$, $AI = 4$, $HC = 4$, $GC = 5$, and $DE + FG = 8$, find the value of DE.

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Ryan Tang

Solution: By Power of a Point, $AD \cdot AE = AI \cdot AH$, and since $AI = CH$ and $AH = CI$, we can manipulate this into $AD \cdot AE = CH \cdot CI = CG \cdot CF$ by Power of a Point again. Plugging our values in, this becomes $3AE = 5CF$.
Now, we set $DE = x$ and $FG = 8 - x$. Our equation becomes $3(3 + x) = 5(5 + 8 - x)$, and solving this yields $\boxed{\textbf{(E) } 7}$.

17. There are 8 crabs and a box arranged in a line, with 5 crabs to the left of the box, and 3 crabs to the right of the box. Every second, a crab neighboring the box jumps into the box and disappears. What is the probability that the rightmost crab is the last one to disappear?

(A) $\frac{15}{128}$ **(B)** $\frac{21}{128}$ **(C)** $\frac{29}{128}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{51}{128}$

Ryan Tang

Solution: We casework based on how many crabs on the right of the box have already jumped in when all the left side crabs disappear.
Suppose when all 5 crabs on the left jump in, the three on the right are still present. The probability is $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$ in this case.
Next, consider when one right crab jumps in before all the left crabs jump in. Consider the string $LLLLL$. There are 5 ways to insert the R crab so that the R appears before the last L. Then, each choice has a $\frac{1}{2}$ chance of happening, resulting in $5 \cdot \left(\frac{1}{2}\right)^6 = \frac{5}{64}$ for our probability in this case.
Lastly, if the right crab is the last one remaining when all the ones on the left disappear, we must insert two R's before the last L appears. There are $\binom{6}{2} = 15$ ways to do so, so our probability in this case is $15 \cdot \left(\frac{1}{2}\right)^7 = \frac{15}{128}$.
Summing these up, our answer is $\frac{1}{32} + \frac{5}{64} + \frac{15}{128} = \boxed{\textbf{(C) } \frac{29}{128}}$.
Note: This problem had by far the least correct to incorrect ratio–only approximately $\frac{1}{6}$ of respondents who answered this question got it correct. Perhaps the large number of respondents who answered the problem incorrectly used the bogus solution disproved below.
Note 2: The answer is not $\frac{3}{8}$ by rearranging $LLLLLRRR$, since after all the crabs in one direction jump in, the probability changes from $\frac{1}{2}$ into 1. As a result, not all cases are equally likely, so this method doesn't work.

18. In a math competition with 100 participants ranked from 1 to 100, Ryan randomly chooses an integer $N > 4$, then randomly interviews 2 of the top N participants. What is the probability that both the first and second-ranked participants are interviewed?

(A) $\frac{1}{2500}$ **(B)** $\frac{1}{1000}$ **(C)** $\frac{1}{500}$ **(D)** $\frac{1}{200}$ **(E)** $\frac{1}{100}$

Ryan Tang

Solution: Given a fixed N, the probability the first and second-ranked participants are interviewed is $\frac{1}{\binom{N}{2}} = \frac{2}{N(N-1)} = \frac{2}{N-1} - \frac{2}{N}$. Thus, as N ranges from 5 to 100, our sum is $\frac{2}{4} - \frac{2}{5} + \frac{2}{5} - \frac{2}{6} + \ldots + \frac{2}{99} - \frac{2}{100} = \frac{12}{25}$. However, we must divide by 96, as each N has a $\frac{1}{96}$ probability of being selected, and this gives us our final answer of $\boxed{\textbf{(D)}\ \frac{1}{200}}$.

19. How many ways are there to rearrange the letters in the word NAGGING so that no G's are adjacent?

(A) 48 **(B)** 60 **(C)** 120 **(D)** 168 **(E)** 240

Wesley Wu

Solution: Consider the possible positions of the three G's. There are ten possible locations of the G, as detailed below. They can also be counted systematically through $(3+2+1)+(2+1)+(1)$, achieved using casework on the positions of the first and second G.

G_G_G__	G___G_G
G_G__G_	_G_G_G_
G_G___G	_G_G__G
G__G_G_	_G__G_G
G__G__G	__G_G_G

The other four letters can be placed into the blanks in $\frac{4!}{2!} = 12$ ways (to account for the duplicate N), so the answer is $12 \cdot 10 = \boxed{\textbf{(C)}\ 120}$.

Alternate Solution: We proceed with complementary counting. To count the number of cases where two G's are together, we treat them as a combined letter. We compute the number of ways to arrange $NA[GG]ING$, which is $\frac{6!}{2!} = 360$. However, this overcounts the case where there are 3 G's in a row. This is counted by the number of ways to arrange $NA[GGG]IN$, which is $\frac{5!}{2!} = 60$. Thus, the number of cases with adjacent G's is $360 - 60 = 300$, and the answer is $\frac{7!}{3!2!} - 300 = \boxed{\textbf{(C)}\ 120}$.

20. How many ordered pairs of integers (m, n) satisfy the equation

$$\sqrt{m} + \sqrt{m+100} = \sqrt{n}?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** Infinitely many

Ryan Tang

Solution: Note that for numbers x, y, z to satisfy $\sqrt{x} + \sqrt{y} = \sqrt{z}$, we must have $x = a^2k$, $y = b^2k$, and $z = c^2k$, where k is square-free. Then, since $x = m$, $y = m + 100$, and $z = n$, we have $y = x = k(b^2 - a^2) = k(b-a)(b+a) = 100$. Furthermore, $b-a$ and $b+a$ should have the same parity, meaning that either both are odd or both are even. This means that k must be odd or divisible by 4, so the only possible values for k are 1 and 5.
If $k = 1$, the only possible cases are $\{b - a, b + a\} = \{2, 50\}, \{10, 10\}$, which result in $\{m, m + 100, n\} = \{576, 676, 2500\}$ and $\{0, 100, 100\}$.
If $k = 5$, the only possible case is $\{b-a, b+a\} = \{2, 10\}$, which gives $\{m, m+100, n\} = \{80, 180, 500\}$.
There are $\boxed{\textbf{(D)}\ 3}$ cases.

Alternate Solution: We square both sides to get $2m + 100 + 2\sqrt{m(m + 100)} = n$. Since everything is an integer (m cannot be a half-integer), $m(m+100) = m^2+100m = (m+50)^2-2500$ must be a perfect square. Then, if we let $j = m + 50$, we must have $j^2 - k^2 = (j - k)(j + k) = 2500$. Both factors must be even, so our possible choices for $\{j - k, j + k\}$ are $\{2, 1250\}, \{10, 250\}$, or $\{50, 50\}$, which lead to $\{j, k\}$ being $\{626, 624\}, \{130, 120\}$, or $\{50, 0\}$. These three cases result in the same three ordered pairs.
Note: This problem had a correct to incorrect ratio of almost exactly $1 : 1$.

21. For how many integers n between 1 and 100, inclusive, is $\frac{n!^5}{5^n}$ an integer?

(A) 25 **(B)** 30 **(C)** 50 **(D)** 75 **(E)** 80

Ryan Tang

Solution: For $1 \le n < 25$, $n!$ has $\lfloor \frac{n}{5} \rfloor$ factors of 5, so $n!^5$ has $5\lfloor \frac{n}{5} \rfloor$ factors of 5, while 5^n has n factors of 5. For this quotient to be an integer, 5 must divide into n. This results in $n = 5, 10, 15, 20$, with 4 solutions.
For $n \ge 25$, $n!^5$ has $5(\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{25} \rfloor + \cdots)$ factors of 5, while 5^n has n factors of 5. Note that

$$5\left(\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \cdots\right) \ge 5\left(\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor\right) \ge 5\left(1 + \left\lfloor \frac{n}{5} \right\rfloor\right) \ge 5\left(\frac{n-4}{5} + 1\right) = n + 1 > n,$$

so the quotient is always an integer. This case has $100 - 25 + 1 = 76$ solutions.
Thus, the answer is $4 + 76 = \boxed{\textbf{(E)}\ 80}$.

22. Bruce has 55 dice, each numbered 1 through 10. Each face of every die is weighted so that the probability of rolling n is directly proportional to n. Patrick rolls each of the die, and he allows Patrick to reroll each die once if he wants. Patrick wants to maximize the total sum of the dice and plays optimally. What is the expected value of this sum?

(A) 385 **(B)** 437 **(C)** 441 **(D)** 464 **(E)** 505

Bruce Shu

Solution: The probability of each n occurring is $\frac{n}{55}$. Thus, the expected value of a roll is

$$1 \cdot \frac{1}{55} + 2 \cdot \frac{2}{55} + 3 \cdot \frac{3}{55} + \cdots + 10 \cdot \frac{10}{55} = \frac{1^2 + 2^2 + \cdots + 10^2}{55} = \frac{10(11)(21)}{6 \cdot 55} = 7.$$

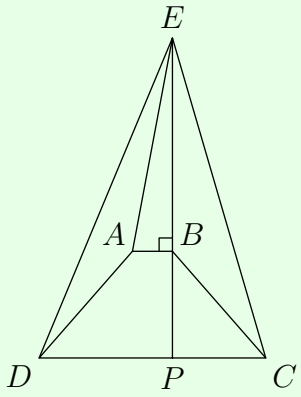
The strategy is to reroll if the number is less than 7, which has a $\frac{1}{55} + \frac{2}{55} + \cdots + \frac{6}{55} = \frac{6(7)}{55 \cdot 2} = \frac{21}{55}$ chance of happening. This makes the finalexpected value $\frac{21}{55} \cdot 7 + \frac{7}{55} \cdot 7 + \frac{8}{55} \cdot 8 + \cdots + \frac{10}{55} \cdot 10 = \frac{147+49+64+81+100}{55} = \frac{441}{55}$, so the answer is $55 \cdot \frac{441}{55} = \boxed{\textbf{(C)}\ 441}$ by linearity of expectation.
Note: If it is unclear why we should reroll below the expected value, consider a separate problem with a $\frac{99}{100}$ chance of rolling a 1 and a $\frac{1}{100}$ chance of rolling a 99.

23. Isosceles trapezoid $ABCD$ has $AB \parallel CD$ and $AB = 3$. Triangle ABE is drawn outside the trapezoid with $\angle ABE = 90^\circ$. If $EC = 25$, $ED = 26$, and the area of pentagon $AEBCD$ is 104, compute the area of triangle BDE.

(A) 54 **(B)** 76 **(C)** 80 **(D)** 104 **(E)** 204

Wesley Wu

Solution: Below is a diagram with the height from B to CD dropped.



Since $ABCD$ is isosceles, $DP = CP + AB = CP + 3$. Then, applying Pythagoras on DPE and CPE yields $26^2 - (CP+3)^2 = 25^2 - CP^2$, which we can solve for to get $CP = 7$, $DP = 10$, and $EP = 24$. The area of BDE can be expressed as $\frac{1}{2} \cdot EB \cdot DP = 5EB$. Let $n = EB$. Then, the area of EAB is $\frac{3n}{2}$, and the area of $ABCD$ is $10BP = 10(24 - n) = 240 - 10n$. Then, $240 - 10n + \frac{3n}{2} = 104$, which has solution $n = 16$. Therefore, our answer is $\boxed{\textbf{(C)}\ 80}$.

24. How many ordered pairs (a, b) are there to the equation

$$\gcd(a,b)^3 + 90 \gcd(a,b) = (\gcd(a,b) + 1)(b - a)$$

if $1 \le a \le 500$?

(A) 42 **(B)** 43 **(C)** 70 **(D)** 98 **(E)** 129

Ryan Tang

Solution: Let $d = \gcd(a, b)$ so that $a = dx$ and $b = dy$ with $\gcd(x, y) = 1$. Since both sides are positive, we must have $y > x$, too. The equation becomes

$$d^3 + 90d = d(d+1)(y-x) \Longrightarrow d^2 + 90 = (d+1)(y-x).$$

Hence,

$$y - x = \frac{d^2+90}{d+1} = d - 1 + \frac{91}{d+1},$$

so $d + 1 \mid 91$. Then, $d + 1$ must be among $\{1, 7, 13, 91\}$ and d must be positive, d must be $6, 12$, or 90. We can calculate that k is equal to $18, 18$, and 90, respectively.

For each d, we must count the number of positive integers x with $dx \leq 500$ and $\gcd(x, k) = 1$, which fixes y as well. Therefore, the number of solutions is the number of positive integers $x \leq \left\lfloor \frac{500}{d} \right\rfloor$ with $\gcd(x, k) = 1$.

For $d = 6$, of every $k = 18$ numbers, there must be $\phi(k) = 6$ x satisfying the condition. There are $\left\lfloor \frac{500}{6} \right\rfloor = 83$ unfiltered choices of x, and there are $\left\lfloor \frac{83}{18} \right\rfloor = 4$ periods of 18, resulting in $4 \cdot 6 = 24$ cases. The remaining numbers to consider are $72 < x \leq 83$, of which only $x = 73, 77, 79, 83$ are coprime to k. Adding these gives a total count of 28 for this case.

For $d = 12$, there must be 6 x satisfying the condition every 18 numbers still. There are $\left\lfloor \frac{500}{12} \right\rfloor = 41$ unfiltered choices of x, and there are $\left\lfloor \frac{41}{18} \right\rfloor = 2$ periods of 18, resulting in $2 \cdot 6 = 12$ cases. The remaining numbers are $36 < x \leq 41$, and only $x = 37, 41$ work here, for a total of 14 numbers for this case.

Lastly, for $d = 90$, there are only $\left\lfloor \frac{500}{90} \right\rfloor = 5$ original choices for x. The only x satisfying $\gcd(x, 90) = 1$ is $x = 1$, so there is only one case here.

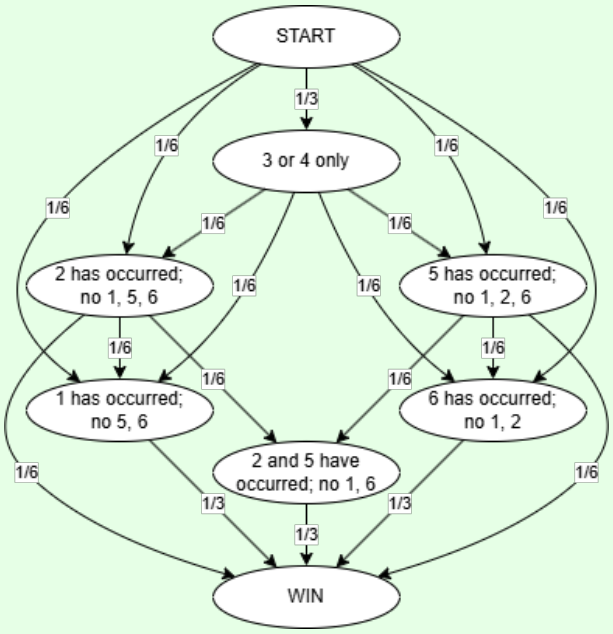
Summing up these results in $28 + 14 + 1 = \boxed{\textbf{(B)}\ 43}$.

25. A fair six-sided die is rolled repeatedly. Let m_k and M_k be the minimum and maximum rolls, respectively, after k rolls of the die. Let n be the least possible n such that $M_n - m_n \geq 4$. Find the expected value of n.

(A) $\frac{7}{2}$ **(B)** 4 **(C)** 5 **(D)** $\frac{23}{4}$ **(E)** 6

Patrick Sun

Solution: Below is a diagram of all the states and their connections to each other; self-connections are omitted for clarity.


START
1/3
3 or 4 only
1/6
2 has occurred; no 1, 5, 6
5 has occurred; no 1, 2, 6
1 has occurred; no 5, 6
6 has occurred; no 1, 2
2 and 5 have occurred; no 1, 6
WIN

Note that the two sides are essentially symmetric. Furthermore, the branches from START and "3 or 4 only" are equivalent; hence, we assign the five rows (excluding START) values A, B, C, D, and E from top to bottom, and the value of START happens to be equal to A. These values count the expected number of moves to travel from those states to the WIN condition–this means that $E = 0$. Now, we consider each state separately.

We have the equations

$$\begin{cases} E = 0 \\ D = \frac{2}{3}D + \frac{1}{3}E + 1 \\ C = \frac{2}{3}C + \frac{1}{3}E + 1 \\ B = \frac{1}{2}B + \frac{1}{6}C + \frac{1}{6}D + \frac{1}{6}E + 1 \\ A = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C + 1 \\ X = A \end{cases}$$

Solving, we find that

$$D = 3, C = 3, B = 4, \text{ and } A = X = \boxed{\textbf{(C)}\ 5}.$$